Lecture 7

Spatial point processes

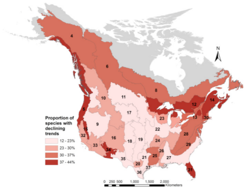
Sara Martino, Dept. of Mathematical Science, NTNU,
Janine Illian,
Jafet Belmont, University of Glasgow

March 7, 2026

Recall: Types of spatial data

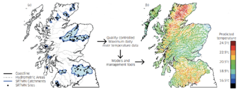
We can distinguish three types of spatial data structures

Areal data



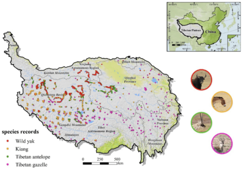
Map of bird conservation regions (BCRs) showing the proportion of bird species within each region showing a declining trend

Geostatistical data



Scotland river temperature monitoring network

Point-referenced data



Occurrence records of four ungulate species in the Tibet,

Types of spatial data

Recall:

Discrete space:

- Data on a spatial grid (areal data)

Continuous space:

- Geostatistical (geo-referenced) data
- **Spatial point data**

Model components are used to reflect spatial dependence structures in discrete and continuous space.

Continuous space: spatial point patterns

- Locations of objects/events in space (typically 2D)
- Examples: tree locations, animal groups, earthquakes

Observed response(s): x,y coordinates (sometimes also marks)

Point patterns vs. geostatistical data

Point patterns:

- Data format: x,y coordinates
- Optional: marks
- Aim: model locations as random

Geostatistical data:

- Data format: x,y coordinates
- Measurements mandatory
- Aim: model continuous process at measured locations

spatial point processes – what are they?

models of spatial patterns:

$\Rightarrow$ modelling **locations and properties ("marks")** of objects, events, individuals in space and time

spatial point processes – what are they?

models of spatial patterns:

⇒ modelling **locations and properties (“marks”)** of objects, events, individuals in space and time

aim: understanding mechanisms
that generated the pattern

spatial point processes – what are they?

models of spatial patterns:

⇒ modelling **locations and properties ("marks")** of objects, events, individuals in space and time

aim: understanding mechanisms that generated the pattern

- examples:
  - cancer cells
  - plants or animals
  - earthquakes
  - terrorist attacks

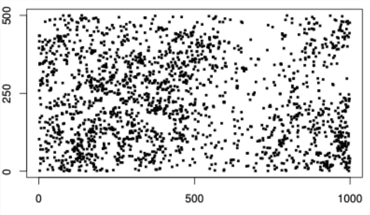
spatial point processes – what are they?

models of spatial patterns:

⇒ modelling **locations and properties (“marks”)** of objects, events, individuals in space and time

aim: understanding mechanisms that generated the pattern

- examples:
  - cancer cells
  - plants or animals
  - earthquakes
  - terrorist attacks



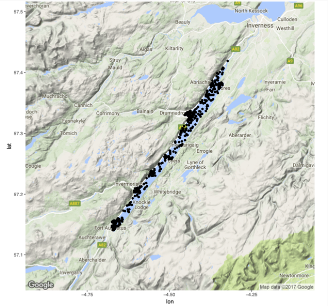
spatial point processes – what are they?

models of spatial patterns:

⇒ modelling **locations and properties (“marks”)** of objects, events, individuals in space and time

aim: understanding mechanisms that generated the pattern

- examples:
    - cancer cells
    - plants or animals
    - earthquakes
    - terrorist attacks



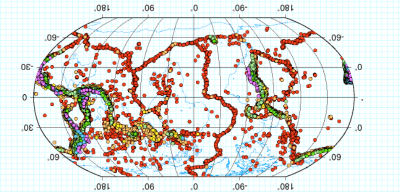
spatial point processes – what are they?

models of spatial patterns:

⇒ modelling **locations and properties (“marks”)** of objects, events, individuals in space and time

aim: understanding mechanisms that generated the pattern

- examples:
    - cancer cells
    - plants or animals
    - earthquakes
    - terrorist attacks

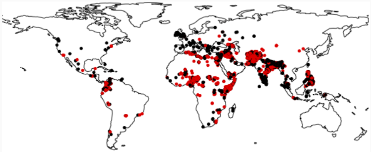


spatial point processes – what are they?

models of spatial patterns:

⇒ modelling **locations and properties ("marks")** of objects, events, individuals in space and time

aim: understanding mechanisms that generated the pattern



- examples:
  - cancer cells
  - plants or animals
  - earthquakes
  - terrorist attacks

spatial point processes – what are they?

models of spatial patterns:

⇒ modelling **locations and properties (“marks”)** of objects, events, individuals in space and time

aim: understanding mechanisms that generated the pattern

- examples:
  - cancer cells
  - plants or animals
  - earthquakes
  - terrorist attacks

spatial point processes – why are they complex?

Point patterns can be modelled by a **point process** N

- assigns a count of points to every subset in W; a **measure**; point process N is a **random (counting) measure**

spatial point processes – why are they complex?

Point patterns can be modelled by a **point process** N

- assigns a count of points to every subset in W; a **measure**; point process N is a **random (counting) measure**

$\Rightarrow$ mathematically complicated statistical models

spatial point processes – why are they complex?

Point patterns can be modelled by a **point process** N

- assigns a count of points to every subset in W; a **measure**; point process N is a **random (counting) measure**

⇒ mathematically complicated statistical models

⇒ spatial models – computationally expensive

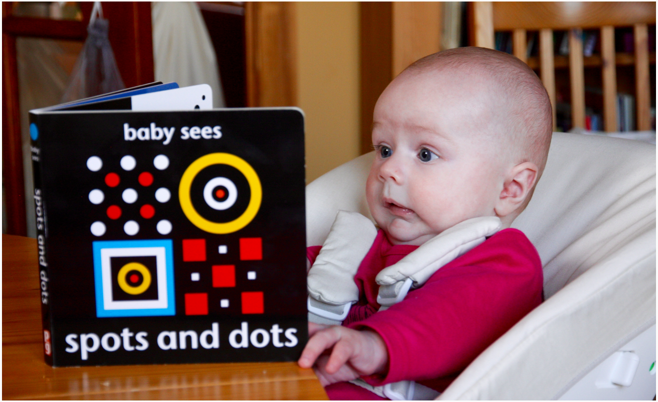
spatial point processes – why are they complex?

Point patterns can be modelled by a **point process** N

- assigns a count of points to every subset in W; a **measure**; point process N is a **random (counting) measure**

$\Rightarrow$ mathematically complicated statistical models

$\Rightarrow$ spatial models – computationally expensive



spatial point processes – and so?

using `R-INLA` and `inlabru` we have been able to develop methodology that

- is practically relevant and realistically complex,
- accounts for reality of data collection and
- is accessible to users

spatial point processes – and so?

using `R-INLA` and `inlabru` we have been able to develop methodology that

- is practically relevant and realistically complex,
- accounts for reality of data collection and
- is accessible to users

$\rightsquigarrow$ making the very theoretic point processes relevant in practice

spatial point processes – and so?

using `R-INLA` and `inlabru` we have been able to develop methodology that

- is practically relevant and realistically complex,
- accounts for reality of data collection and
- is accessible to users

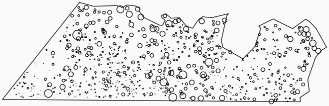
$\rightsquigarrow$ making the very theoretic point processes relevant in practice
$\rightsquigarrow$ some methodology that has been developed for other data structures is not always available for point processes (yet) – e.g. model comparison is difficult (see lecture 10)

marked patterns

- analysis of **marked point patterns** is more interesting
- relevance: provides deeper insight into the processes that are causing the pattern than an analysis of unmarked point patterns
- marks may be
  - **qualitative**, i.e. the pattern is multivariate and consists of several types of points, e.g. different species, ages and size classes, or
  - **quantitative**, i.e. continuous variates, or vectors of variates or even stochastic processes

marked patterns

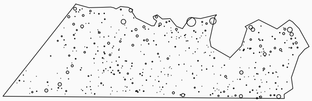
- analysis of **marked point patterns** is more interesting
- relevance: provides deeper insight into the processes that are causing the pattern than an analysis of unmarked point patterns
- marks may be
  - **qualitative**, i.e. the pattern is multivariate and consists of several types of points, e.g. different species, ages and size classes, or
  - **quantitative**, i.e. continuous variates, or vectors of variates or even stochastic processes



locations of eucalyptus trees in a koala reserve; diameters of the circles reflect the palatability of the trees' leaves

marked patterns

- analysis of **marked point patterns** is more interesting
- relevance: provides deeper insight into the processes that are causing the pattern than an analysis of unmarked point patterns
- marks may be
  - **qualitative**, i.e. the pattern is multivariate and consists of several types of points, e.g. different species, ages and size classes, or
  - **quantitative**, i.e. continuous variates, or vectors of variates or even stochastic processes



diameters of the circles reflect the number of times a koala was observed on a tree in a given time period

a model for spatial patterns...

question:

- are the daisies randomly distributed in the lawn of our garden???

issues

- what do we mean by "random"? formal description?
- what if they are not random?
- how should we describe and model non-random patterns?

Poisson point processes

What do we mean by “random”?

Poisson point processes

What do we mean by “random”?

more formally: **complete spatial randomness CSR**:

Poisson point processes

What do we mean by “random”?

more formally: **complete spatial randomness CSR**:

the **homogeneous Poisson process** is a model for CSR:

Poisson point processes

What do we mean by “random”?

more formally: **complete spatial randomness CSR**:

the **homogeneous Poisson process** is a model for CSR:

characterised by two fundamental properties:

Poisson point processes

What do we mean by “random”?

more formally: **complete spatial randomness CSR**:

the **homogeneous Poisson process** is a model for CSR:

characterised by two fundamental properties:

(1) *Poisson distribution of point counts:* number of points of in any set A follows a Poisson distribution with mean $\lambda \, \|A\|$

Poisson point processes

What do we mean by "random"?

more formally: **complete spatial randomness CSR**:

the **homogeneous Poisson process** is a model for CSR:

characterised by two fundamental properties:

(1) *Poisson distribution of point counts:* number of points of in any set A follows a Poisson distribution with mean $\lambda \, \|A\|$

(2) *Independent scattering:* number of points of in k disjoint sets are independent of each other

Poisson point processes

the intensity:

- λ the *intensity* or *point density*, of the homogeneous Poisson process

Poisson point processes

the intensity:

- λ the *intensity* or *point density*, of the homogeneous Poisson process describes the mean number of points per unit area

Poisson point processes

the intensity:

- λ the *intensity* or *point density*, of the homogeneous Poisson process describes the mean number of points per unit area
- $\lambda \cdot \|A\| = \mathbb{E}(N(A)) \quad$ for all sets A

Poisson process – more interesting models...

the properties can be generalised to yield more complex models:

Poisson process – more interesting models...

the properties can be generalised to yield more complex models:

(1) the intensity can vary with location

Poisson process – more interesting models...

the properties can be generalised to yield more complex models:

(1) the intensity can vary with location $\rightsquigarrow \lambda(x)$

inhomogeneous Poisson process or

Poisson process – more interesting models...

the properties can be generalised to yield more complex models:

(1) the intensity can vary with location $\rightsquigarrow \lambda(x)$

inhomogeneous Poisson process or

(2) can be random (i.e. not fixed)

Poisson process – more interesting models...

the properties can be generalised to yield more complex models:
(1) the intensity can vary with location $\rightsquigarrow \lambda(x)$
inhomogeneous Poisson process or

(2) can be random (i.e. not fixed) $\rightsquigarrow \Lambda(x)$, i.e. a random field (see geostatistics!)
Cox process or

Poisson process – more interesting models...

the properties can be generalised to yield more complex models:
(1) the intensity can vary with location $\rightsquigarrow \lambda(x)$
inhomogeneous Poisson process or

(2) can be random (i.e. not fixed) $\rightsquigarrow \Lambda(x)$, i.e. a random field (see geostatistics!)
Cox process or

(3) non-independence: interaction among the points
Gibbs process [not discussed here]

Point process models

constant intensity λ of the homogeneous Poisson process is replaced by an intensity function $\lambda(x)$

Point process models

constant intensity λ of the homogeneous Poisson process is replaced by an intensity function $\lambda(x)$
$\rightsquigarrow$ value varies with the location x

Point process models

constant intensity λ of the homogeneous Poisson process is replaced by an intensity function $\lambda(x)$
$\rightsquigarrow$ value varies with the location x
properties of the inhomogeneous Poisson process;

Point process models

constant intensity λ of the homogeneous Poisson process is replaced by an intensity function $\lambda(x)$
$\rightsquigarrow$ value varies with the location x
properties of the inhomogeneous Poisson process;
fundamental property (1) of the homogeneous Poisson process is generalised, whereas (2) remains unchanged.

Point process models

constant intensity λ of the homogeneous Poisson process is replaced by an intensity function $\lambda(x)$
$\rightsquigarrow$ value varies with the location x
properties of the inhomogeneous Poisson process;
fundamental property (1) of the homogeneous Poisson process is generalised, whereas (2) remains unchanged.

(1) *Poisson distribution of point counts*. number of points N in any set B has a Poisson distribution with mean $\int_B \lambda(x)\mathrm{d}x$
(2) *Independent scattering.* The random numbers of points of N in k disjoint sets are independent random variables, for arbitrary k.

Point process models – Cox processes

Cox process:

- intensity function is replaced by a **random field** $\Lambda(x)$ with non-negative values

Point process models – Cox processes

Cox process:

- intensity function is replaced by a **random field** $\Lambda(x)$ with non-negative values
- realisations of this random field are functions in space
- these are intensity functions of inhomogeneous Poisson processes

Point process models – Cox processes

Cox process:

- intensity function is replaced by a **random field** $\Lambda(x)$ with non-negative values
- realisations of this random field are functions in space
- these are intensity functions of inhomogeneous Poisson processes

Point process models – Cox processes

Cox process:

- intensity function is replaced by a **random field** $\Lambda(x)$ with non-negative values
- realisations of this random field are functions in space
- these are intensity functions of inhomogeneous Poisson processes

example:

log-Gaussian Cox process (LCP):

$$\log(\Lambda) = Z(s),$$

where $Z(.)$ is a **Gaussian random field**

Cox processes are doubly-stochastic processes

Point process models – Cox processes

Cox process:

- intensity function is replaced by a **random field** $\Lambda(x)$ with non-negative values
- realisations of this random field are functions in space
- these are intensity functions of inhomogeneous Poisson processes

example:

log-Gaussian Cox process (LCP):

$$\log(\Lambda) = Z(s),$$

where $Z(.)$ is a **Gaussian random field**

Cox processes are doubly-stochastic processes
$\rightsquigarrow$ flexible, but hard to fit!

The LGCP Model

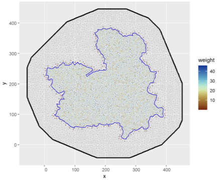
$$p(\mathbf{y}|\lambda) \propto \exp\left(-\int_{\Omega} \lambda(\mathbf{s})d\mathbf{s}\right) \prod_{i=1}^{n} \lambda(\mathbf{s}_i)$$

$$\eta(s) = \log(\lambda(s)) = \boxed{\beta_0} + \boxed{x(s)} + \boxed{\omega(s)}$$

The code

```
# define model component
cmp = ~ Intercept(1) + elev(elev_raster, model = "linear") +
  space(geometry, model = spde_model)


# define model predictor
eta  = geometry ~ Intercept +  elev + space


# build the observation model
lik = bru_obs("cp",
              formula = eta,
              data = pp,
              ips = ips)


# fit the model
fit = bru(cmp, lik)
```

```
# The mesh
mesh = fm_mesh_2d(boundary = region,
                  max.edge = c(5, 10),
                  cutoff = 4, crs = NA)

# The SPDE model
spde_model =  inla.spde2.pcmatern(mesh,
                                  prior.sigma = c(1, 0.5),
                                  prior.range = c(100, 0.5))

# build integration scheme
ips = fm_int(mesh,
             samplers = region)
```



Point processes in `inlabru`

The LGCP Model

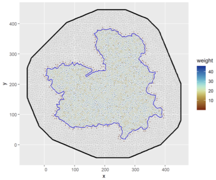
$$p(\mathbf{y}|\lambda) \propto \exp\left(-\int_\Omega \lambda(\mathbf{s})d\mathbf{s}\right)\prod_{i=1}^{n}\lambda(\mathbf{s}_i)$$

$$\boxed{\eta(s)} = \log(\lambda(s)) = \boxed{\beta_0 + x(s) + \omega(s)}$$

The code

```r
# define model component
cmp = ~ Intercept(1) + elev(elev_raster, model = "linear") +
  space(geometry, model = spde_model)


# define model predictor
eta  = geometry ~ Intercept +  elev + space


# build the observation model
lik = bru_obs("cp",
              formula = eta,
              data = pp,
              ips = ips)


# fit the model
fit = bru(cmp, lik)
```

```r
# The mesh
mesh = fm_mesh_2d(boundary = region,
                  max.edge = c(5, 10),
                  cutoff = 4, crs = NA)

# The SPDE model
spde_model =  inla.spde2.pcmatern(mesh,
                                  prior.sigma = c(1, 0.5),
                                  prior.range = c(100, 0.5))

# build integration scheme
ips = fm_int(mesh,
             samplers = region)
```



Point processes in inlabru

The LGCP Model

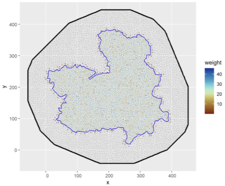
$$\boxed{p(\mathbf{y}|\lambda)} \propto \exp\left(-\int_\Omega \lambda(\mathbf{s})\mathrm{d}\mathbf{s}\right)\prod_{i=1}^{n}\lambda(\mathbf{s}_i)$$

$$\eta(s) = \log(\lambda(s)) = \beta_0 + x(s) + \omega(s)$$

The code

```
# define model component
cmp = ~ Intercept(1) + elev(elev_raster, model = "linear") +
  space(geometry, model = spde_model)


# define model predictor
eta  = geometry ~ Intercept +  elev + space


# build the observation model
lik = bru_obs("cp",
              formula = eta,
              data = pp,
              ips = ips)


# fit the model
fit = bru(cmp, lik)
```

```
# The mesh
mesh = fm_mesh_2d(boundary = region,
                  max.edge = c(5, 10),
                  cutoff = 4, crs = NA)

# The SPDE model
spde_model =  inla.spde2.pcmatern(mesh,
                                  prior.sigma = c(1, 0.5),
                                  prior.range = c(100, 0.5))

# build integration scheme
ips = fm_int(mesh,
             samplers = region)
```



Point processes in inlabru

The LGCP Model

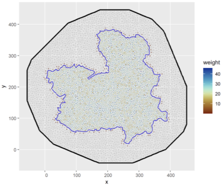
$$p(\mathbf{y}|\lambda) \propto \exp\left(-\int_{\Omega} \lambda(\mathbf{s})\mathrm{d}\mathbf{s}\right) \prod_{i=1}^{n} \lambda(\mathbf{s}_i)$$

$$\eta(s) = \log(\lambda(s)) = \beta_0 + x(s) + \omega(s)$$

The code

```
# define model component
cmp = ~ Intercept(1) + elev(elev_raster, model = "linear") +
  space(geometry, model = spde_model)


# define model predictor
eta  = geometry ~ Intercept +  elev + space


# build the observation model
lik = bru_obs("cp",
              formula = eta,
              data = pp,
              ips = ips)


# fit the model
fit = bru(cmp, lik)
```

```
# The mesh
mesh = fm_mesh_2d(boundary = region,
                  max.edge = c(5, 10),
                  cutoff = 4, crs = NA)

# The SPDE model
spde_model =  inla.spde2.pcmatern(mesh,
                                  prior.sigma = c(1, 0.5),
                                  prior.range = c(100, 0.5))

# build integration scheme
ips = fm_int(mesh,
             samplers = region)
```



Cox processes - why are they so versatile?

Consider a standard Poisson model (non-spatial):

- assumes Poisson distribution of counts
- in practice this does not always hold (overdispersion)

Cox processes - why are they so versatile?

Consider a standard Poisson model (non-spatial):

- assumes Poisson distribution of counts
- in practice this does not always hold (overdispersion)

Cox processes - why are they so versatile?

Consider a standard Poisson model (non-spatial):

- assumes Poisson distribution of counts
- in practice this does not always hold (overdispersion) make rate parameter λ random – accounts for over dispersion

In spatial modelling:
Cox process – the intensity $\Lambda(.)$ is random; when Gaussian field, spatially structured), i.e. smooth in space

Cox processes - why are they so versatile?

Consider a standard Poisson model (non-spatial):

- assumes Poisson distribution of counts
- in practice this does not always hold (overdispersion) make rate parameter λ random – accounts for over dispersion

In spatial modelling:
Cox process – the intensity $\Lambda(.)$ is random; when Gaussian field, spatially structured), i.e. smooth in space
Intuitively:
Can account for additional "spatial over dispersion" that is structured- i.e. patterns that covariates cannot explain

and what about marked point processes?

recall: marked point processes

- also contain information on the object/event/individual represented by the point – one- or several - mark(s)

and what about marked point processes?

recall: marked point processes

- also contain information on the object/event/individual represented by the point – one- or several - mark(s)
- mark can also be modelled – has its own likelihood

and what about marked point processes?

recall: marked point processes

- also contain information on the object/event/individual represented by the point – one- or several - mark(s)
- mark can also be modelled – has its own likelihood
- joint model of mark(s) and point pattern if dependent

and what about marked point processes?

recall: marked point processes

- also contain information on the object/event/individual represented by the point – one- or several - mark(s)
- mark can also be modelled – has its own likelihood
- joint model of mark(s) and point pattern if dependent
- see lecture tomorrow on joint modelling

What happens next

- Practical 5 continued: spatial point process fitting
- homogeneous Poisson process
- inhomogeneous Poisson process
- Log-Gaussian Cox process

Take-home message

- spatial point processes model spatial point patterns
- log Gaussian Cox processes are based on Gaussian random fields
- the SPDE approach allows us – again – to approximate these efficiently and flexibly
- these models are still GMRFs
  $\rightsquigarrow$ computationally efficient representation

Take-home message

- spatial point processes model spatial point patterns
- log Gaussian Cox processes are based on Gaussian random fields
- the SPDE approach allows us – again – to approximate these efficiently and flexibly
- these models are still GMRFs
  $\rightsquigarrow$ computationally efficient representation