Lecture 6

Introduction to spatial modelling – geo-referenced data

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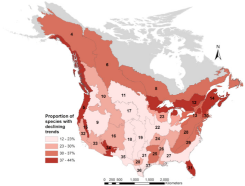
Recall: Types of spatial data

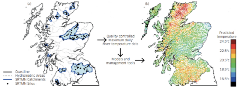
We can distinguish three types of spatial data structures

Areal data



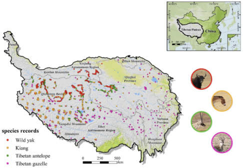
Map of bird conservation regions (BCRs) showing the proportion of bird species within each region showing a declining trend

Geostatistical data



Scotland river temperature monitoring network

Point-referenced data



Occurrence records of four ungulate species in the Tibet,

Types of spatial data

Recall:

Discrete space:

- Data on a spatial grid (areal data)

Continuous space:

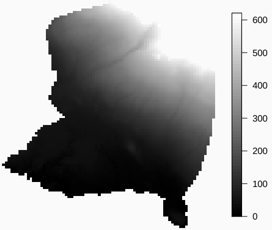
- **Geostatistical (geo-referenced) data**
- Spatial point pattern data

Model components are used to reflect spatial dependence structures in discrete and continuous space.

spatial data – continuous case

recall: geostatistical data

- phenomenon that is continuous in space
- examples: nutrient levels in soil, salinity in the sea, altitude
- measurements in only a finite number of locations



aim: estimate the continuous field

- continuous random variable, a **random field**: function in space with values in continuous space
- **Gaussian random field**
- characterised by mean and covariance (function)

geostatistical data

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  $\rightsquigarrow$ flexibly and efficiently using the **SPDE** approach

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- What makes a function Gaussian...?

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the mesh!

the SPDE approach – more flexible models

- simple models use a simple (regular) gridding approach to approximate the continuous spatial field
- this is easy to implement
- however: this can be
    - **computationally inefficient** and
    - not flexible enough (complicated boundaries or domains)

$\Rightarrow$ use continuously specified finite dimensional Gaussian random fields

$\Rightarrow$ spatial field as solution to a stochastic partial differential equation ("SPDE approach")

continuous specification

- use a **continuous** specification of the random field model:

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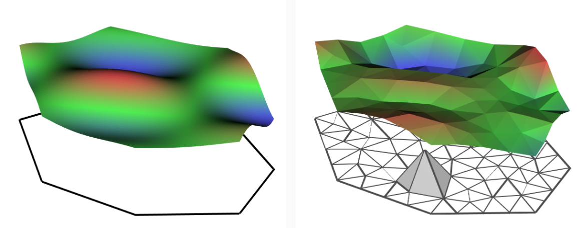
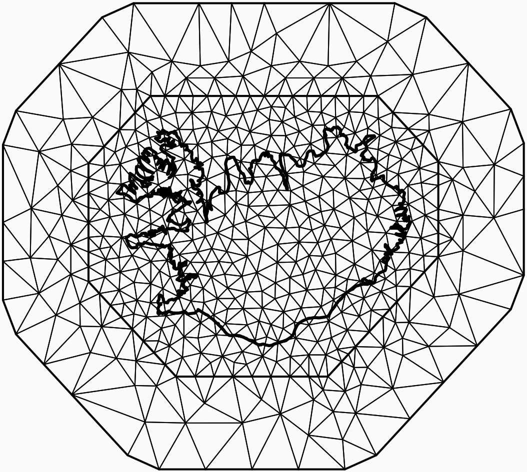
  a finite-dimensional basis function expansion
  in the "mesh" - a triangulation of space



  - e.g. for point processes: no "binning" of the points
- allows computation using the **exact** positions of the points...

A mesh...



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After all those technicalities... here's the important bit: The SPDE approach yields

- more flexible modelling; it is easier to
  - build more general models
  - work with changing observation areas over time
  - work with "funny" observation areas
- we don't need to worry about covariance functions...

Part of the magic: SPDE models are still GMRF!

$\Rightarrow$ we can still use INLA to fit these and it is still fast!

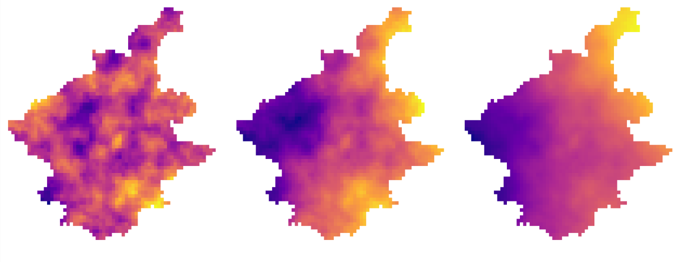
SPDE models

We call spatial Markov models defined on a mesh *SPDE models*.

SPDE models have 3 parts

- a mesh
- a range parameter ρ
- a variance parameter σ^2 (or precision parameter τ)

Different realisations of the same SPDE model with varying range parameter κ.



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Here:

$$y(s)|\eta(s) \sim \text{Binom}(1, p(s))$$

$$\eta(s) = \text{logit}(p(s)) = \beta_0 + \omega(s) + \beta_1 \text{ depth}(s)$$

prior choice for GRFs

priors determine the smoothness of the random field

- if the field is too smooth, spurious significance
- if the field is too wiggly, overfitting

$\Rightarrow$ choice has to be done very carefully

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- believe in some default prior and trust it blindly

all arbitrary...

as a software developer...

options:

1) provide some standard prior distributions

- e.g. a log-gamma distribution; user chooses parameters

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- choose a criterion (e.g. "degree of wigglyness") and penalise violation of that criterion in an automated way

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⇒ BUT: we don't know what the ideal smoothness/wigglyness is...

as a software developer...

here: re-think priors:

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penalise something different: deviation from a base model

aim: make prior choice transparent and problem-driven

Occam's razor: Prefer simpler model until there is enough support for a more complex model.

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- model component is flexible version of a **base model**
- penalise deviation from base model (using Kullback-Leibler divergence, measuring information loss)
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- the maths says that having a prior on this parameterisation will avoid overfitting
- we are penalising unnecessary model complexity
- also: interpretable priors and **flexible modelling**

make prior choice transparent

penalise deviation from a base model in our context...

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The base model:

- model with intercept and covariates, no (or rather a very flat) Gaussian random field

$\Rightarrow$ the approach is conservative about including a random field – penalises overfitting

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⇒ the approach is conservative about including a random field – penalises overfitting

⇒ we set a prior on the spatial scale that we consider overfitting
⇒ we have another prior that reflects how confident we are about this scale

pc prior for range

prior value that reflects the spatial scale that we consider overfitting

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How do we do this?

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Use `inla.spde2.pcmatern` with prior parameters:

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spde_model1 =  inla.spde2.pcmatern(mesh,
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- i.e. we are pretty sure that a range of this size or smaller is overfitting

Take-home message

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- the SPDE approach allows us to approximate these efficiently and flexibly
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- pc-priors are also relevant for spatial models, e.g. to avoid overfitting