Lecture 5

Modelling areal data

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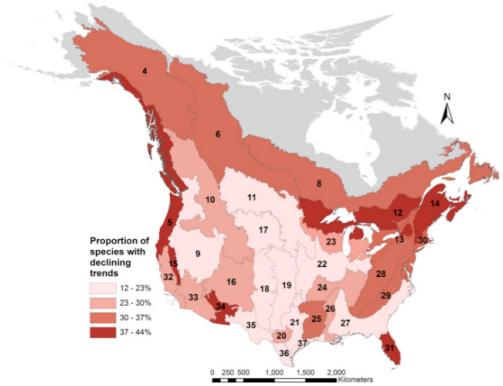
Recall: Types of spatial data

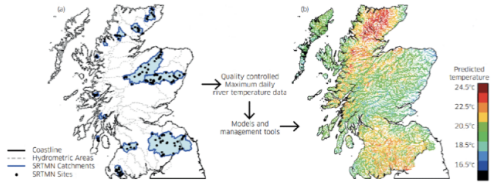
We can distinguish three types of spatial data structures

Areal data



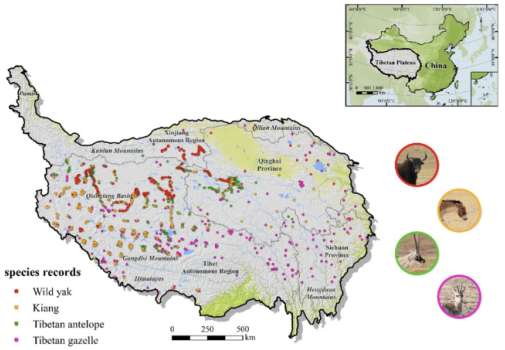
Map of bird conservation regions (BCRs) showing the proportion of bird species within each region showing a declining trend

Geostatistical data



Scotland river temperature monitoring network

Point-referenced data



Occurrence records of four ungulate species in the Tibet,

Types of spatial data

Recall:

Discrete space:

- **Data on a spatial grid (areal data)**

Continuous space:

- Geostatistical (geo-referenced) data
- Spatial point pattern data

Model components are used to reflect spatial dependence structures in discrete and continuous space.

areal data – model component

discrete data in space are related to discrete data in time

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$\rightsquigarrow$ very similar thinking

- neighbourhood structure – neighbours in space
- e.g. we can consider random walk models in space

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- neighbourhood structure – neighbours in space
- e.g. we can consider random walk models in space
- these are special cases of **Gauss Markov random fields** (GMRF)
- recall: latent **Gaussian** models...

large latent Gaussian models

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- real-world applications often involve latent fields with $n = 10^2$–10^5 elements
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- GMRFs provide a computationally efficient representation
- we will see later that these will come in handy even for continuous data (the "SPDE approach")

The Gaussian distribution

A Gaussian distribution $x \sim \mathcal{N}(0, \Sigma)$ is controlled by:

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- The mean vector, which is set equal to 0 here, is the centre of the distribution
- The matrix Σ describes all pairwise covariances $\Sigma_{i,j} = \mathrm{Cov}[x_i, x_j]$
- Analytically tractable and well understood...

Formally

$$\pi(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}x^T\Sigma^{-1}x}$$

Computational Issues

- Σ is typically large and dense
- infeasible for large n
- we need to reduce computational cost

Sparse Matrices

A matrix Q is **sparse** if most of its elements are zero.

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Two Options

1. Force Σ to be sparse $\Rightarrow$ independence...
2. Force $Q = \Sigma^{-1}$ (the precision) to be sparse $\Rightarrow$ conditional independence

Conditional independence

Example: AR(1) Process

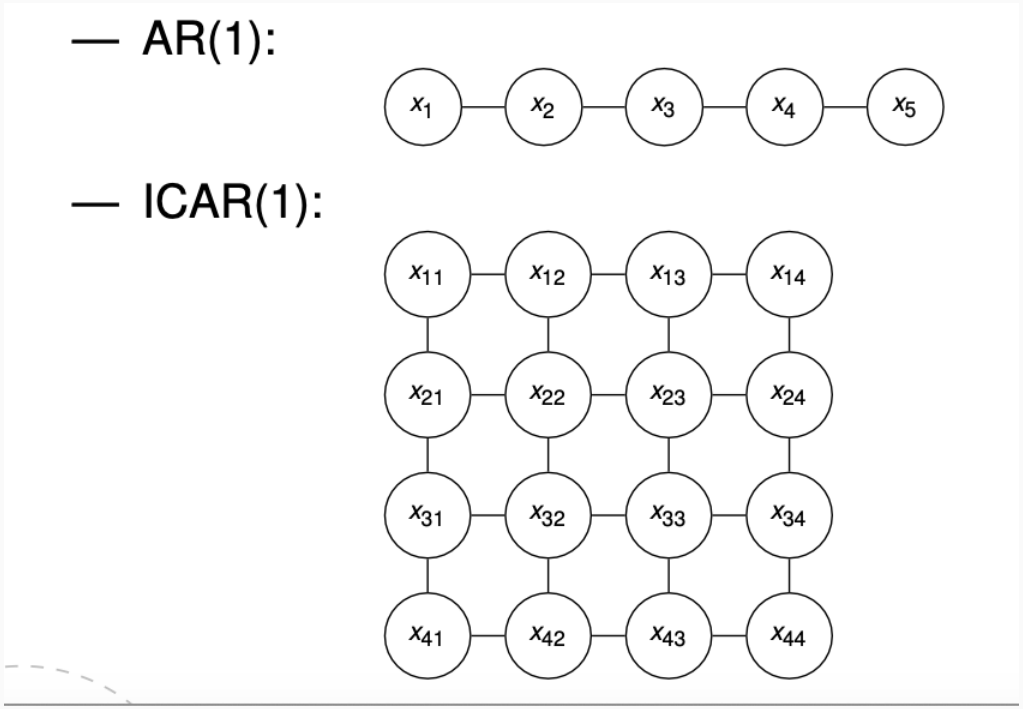
$$x_1 \sim \mathcal{N}\left(0, \frac{1}{1-\phi^2}\right),$$

$x_t = \phi x_{t-1} + \varepsilon_t,\ \varepsilon_t \sim \mathcal{N}(0,1)$

- Σ is dense.
- But Σ^{-1} is tridiagonal (sparse).

Graphical Representation

- Conditional independence structure can be represented via a graph.
- Each node corresponds to a variable x_i.
- Edges correspond to nonzero Q_{ij}.



Definition of a GMRF

A random vector x is a **Gaussian Markov Random Field (GMRF)** on a graph $G = (V, E)$ with mean μ and precision Q if:

$$\pi(x) = \frac{|Q|^{1/2}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(x-\mu)^T Q(x-\mu)\right],$$

and $Q_{ij} \neq 0 \Leftrightarrow \{i, j\} \in E$.

Example: AR(1) process

$$x_1 \sim \mathcal{N}\left(0, \frac{1}{1-\phi^2}\right)$$

$$x_t = \phi x_{t-1} + \varepsilon_t, \quad t > 1, \quad \varepsilon_t \sim \mathcal{N}(0,1)$$

The covariance matrix is dense since all time points are dependent, but the precision matrix Σ^1 is not:

$$\Sigma^{-1} = \begin{bmatrix} 1 & -\phi & 0 & \cdots & 0 \\ -\phi & 1+\phi^2 & -\phi & \cdots & 0 \\ 0 & -\phi & 1+\phi^2 & -\phi & \vdots \\ \vdots & \vdots & -\phi & 1+\phi^2 & -\phi \\ 0 & \cdots & 0 & -\phi & 1 \end{bmatrix}$$

Example: AR(1) process

- the tridiagonal form of Σ^{-1} can be exploited for quick calculations
- we call $Q = \Sigma^{-1}$ the precision matrix.
- key property that causes Σ^{-1} to be sparse – conditional independence

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- we call $Q = \Sigma^{-1}$ the precision matrix.
- key property that causes Σ^{-1} to be sparse – conditional independence
- Can we take advantage of it in other models?

(Informal) definition of a GMRF

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$\rightsquigarrow$ efficient computation

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iCAR or Besag model:

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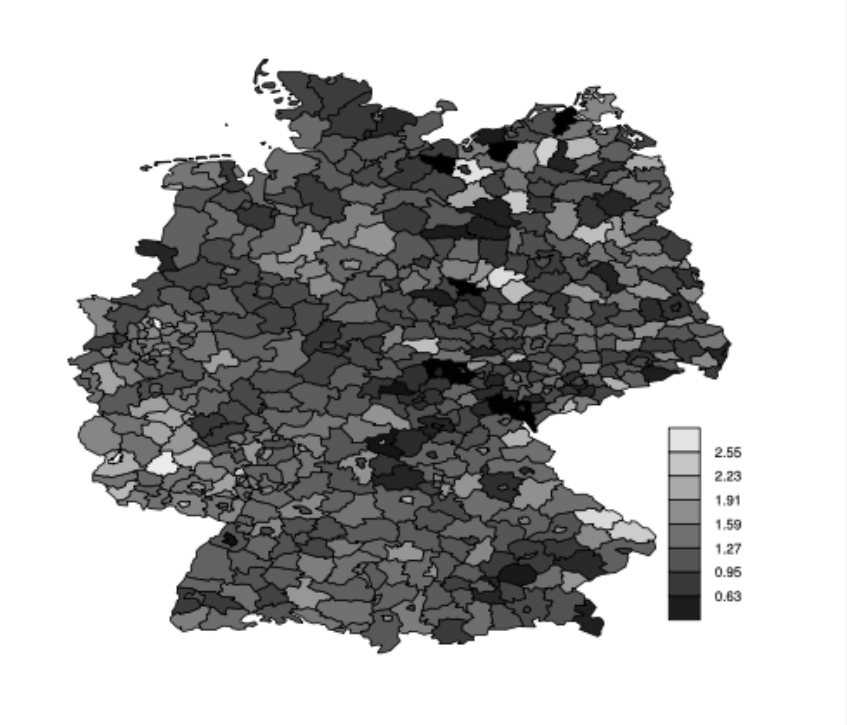

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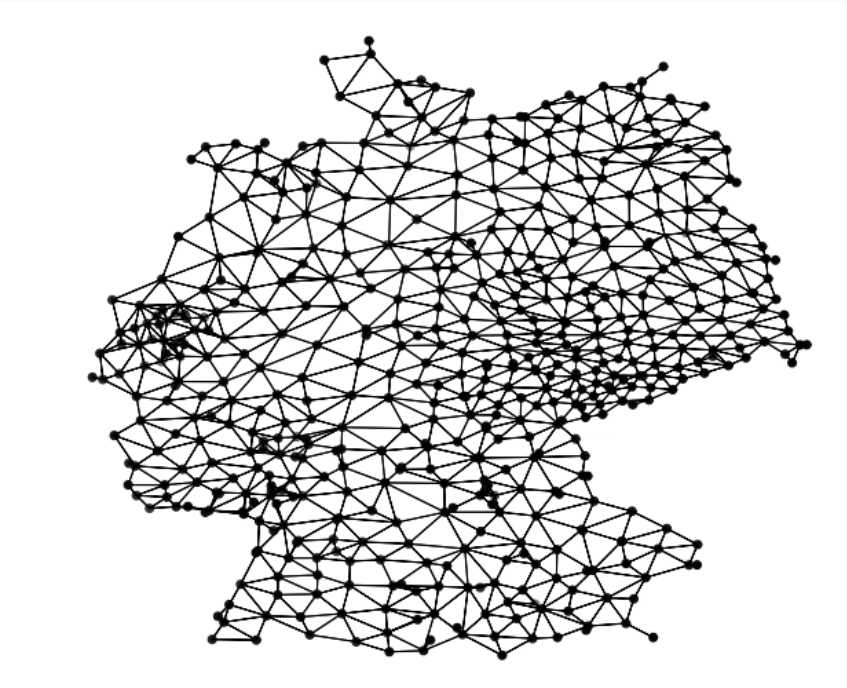

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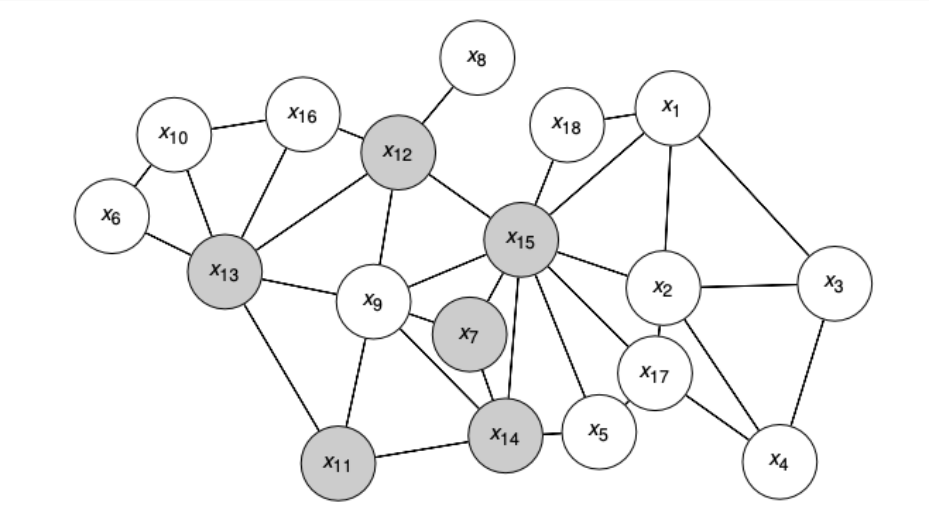

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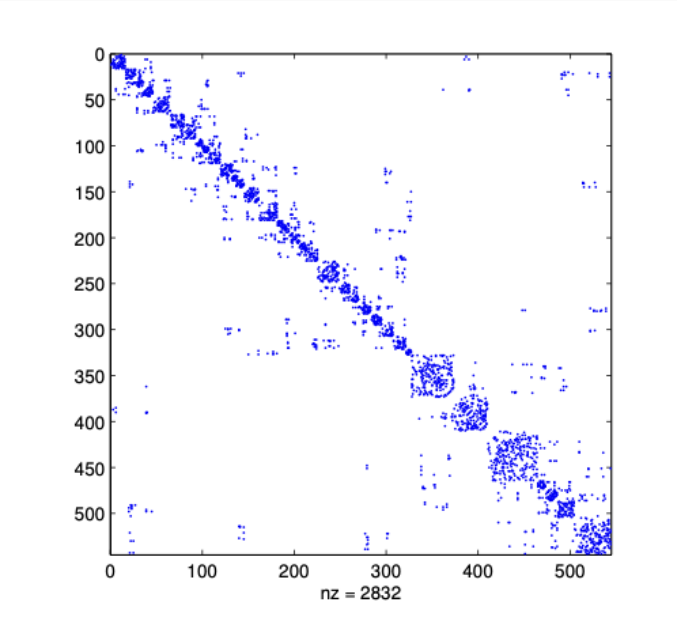
example: larynx cancer data from Germany – precision matrix of the sub graph



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Here – BYM (Besag-York-Mollié) model:

$$y_i|\eta_i \sim \text{Poisson}(E_i\lambda_i)$$

$$\eta_i = \log(\lambda_i) = \beta_0 + \beta_1 pm(10) + \omega_i + z_i$$

The BYM model in `inlabru`

The Model

$$y_i|\eta_t \sim \text{Poisson}(E_i\lambda_i)$$
$$\log(\lambda_i) = \eta_i = \beta_0 + \beta_1 \text{pm10} + u_i + z_i$$

The code

```
# define model component
cmp = ~ Intercept(1) +  pm10(pm10, model = "linear") +
  space(space, model = "besag", graph = Q) +
  iid(space, model = "iid")

# define model predictor
eta =  observed ~ Intercept + space + iid

# build the observation model
lik = bru_obs(formula = formula,
              family = "poisson",
              E = expected,
              data = resp_cases)

# fit the model
fit = bru(cmp, lik)
```

neighbourhood structure

```
library(spdep)
W.nb <- poly2nb(GGHB.IZ,queen = TRUE)
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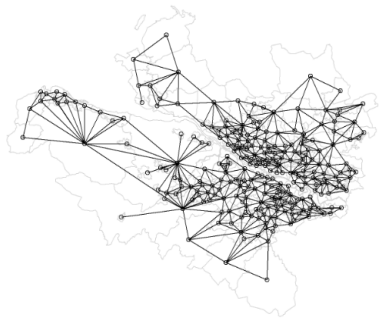

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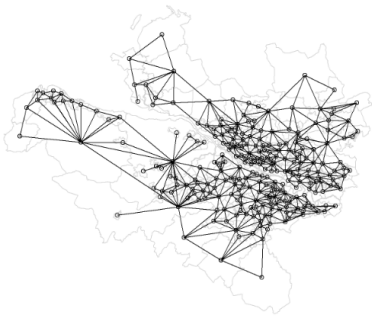

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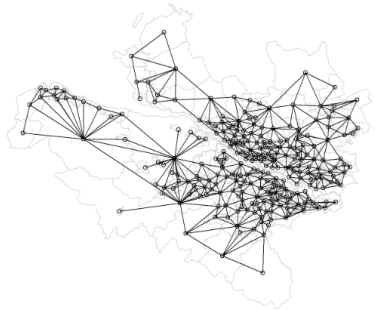

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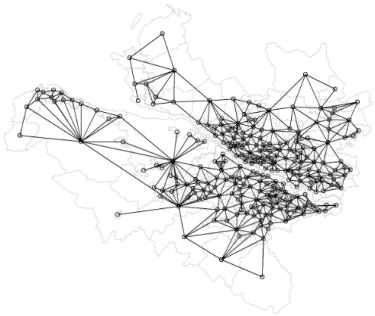

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- look forward to GRMFs in continuously indexed space...