Lecture 10

Multiple likelihood models – joint modelling

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joint models

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- two surveys on the same species using the same sampling approach
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- marked point patterns
- ...

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- **a survey on several species**
- **marked point patterns**
- ...

these are models with more than one likelihood

`inlabru` can be used to fit these models in a straight forward way

recall: marked point patterns

marked point patterns:

- data format: x,y coordinates
  optional: properties of objects represented by the points ("marks")
- aim: model the locations of objects in continuous space
- locations are being modelled and are considered **random**
- marks only take values in locations where there is a point

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a model with **two likelihoods**

joint model...

we have two likelihoods – and two linear predictors

- the first one (e.g. for the point pattern, e.g. of trees)

$$\eta(s) = \beta_1 + \omega(s) + e_1(s),$$

- the second one (e.g. marks, e.g. dbh (diameter at breast height) of trees); depends on the pattern through a joint spatially structured effect

$$\kappa(s) = \beta_2 + \beta_3 \cdot \omega(s) + e_2(s),$$

  where e_1 and e_2 are uncorrelated error terms.

or a slightly more complicated model

- the first one

$$\eta(s) = \beta_1 + \omega_1(s) + e_1(s),$$

- the second one depends on first one through a joint spatially structured effect
- and also has its own spatial structured field $\omega_2(s)$

$$\kappa(s) = \beta_2 + \omega_2(s) + \beta_3 \cdot \omega_1(s) + e_2(s),$$

where e_1 and e_2 are uncorrelated error terms

joint models in inlabru

example in practical – very simple exercise on simulated data:
two likelihoods

- the first one assuming Gaussian response

$$y_1(s) = \beta_1 + \omega(s) + e_1(s),$$

- the second one assuming a Poisson

$$y_2(s) = \beta_2 + \beta_3 \cdot \omega(s) + e_2(s),$$

where e_1 and e_2 are uncorrelated error terms

joint models in inlabru

joint models in inlabru

model components:

```
cmp = ~ -1 +  Intercept1(1) + Intercept2(1) +
  omega1(geometry, model = spde) +
  omega1_copy(geometry, copy = "omega1", fixed = FALSE) +
  omega2(geometry, model = spde)
```

joint models in inlabru

model components:

```
cmp = ~ -1 +  Intercept1(1) + Intercept2(1) +
  omega1(geometry, model = spde) +
  omega1_copy(geometry, copy = "omega1", fixed = FALSE) +
  omega2(geometry, model = spde)
```

note: the "copy feature" – we cannot use `omega1` twice,

we need to create a copy of it, using `copy`

the likelihood and call to bru:

```
lik1 = bru_obs(formula = y ~ Intercept1 + omega1,
            family  = "gaussian",
            data = df1)

lik2 = bru_obs(formula = y ~ Intercept2 + omega1_copy + omega2,
            family = "gaussian",
            data = df2)


res = bru(cmp, lik1, lik2)
```

koala data – marked point pattern data

- study conducted at the Koala Conservation Centre on Phillip Island, near Melbourne, Australia, 1993 - 2004
- $\approx$ 20 koalas present in the reserve at all times throughout study; reserve enclosed by a koala-proof fence
- koalas feed on eucalyptus leaves which are toxic to most animals; koalas have adapted to this

Do the koalas **feed selectively**, i.e. do they choose trees with the least toxic/ most nutritious leaves?

complexity – marked point pattern

marks for each tree:

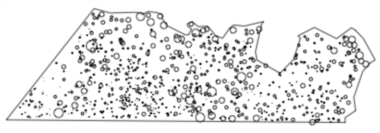
- mark 1: leaf samples taken from each eucalyptus tree and analysed for palatability
- mark 2: tree use by individual koalas collected at monthly intervals between 1993 and March 2004

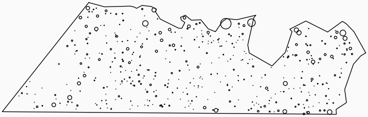
fit joint model to:

- tree locations depend on (unobserved) soil nutrients levels and local clustering
- palatability depends on spatial pattern (through soil nutrients levels)
- koala visitation depends on spatial pattern, palatability

$\Rightarrow$ INLA with multiple likelihoods

The spatial pattern with the leaf marks and the frequency marks





three latent fields...

- for the pattern of trees we have

$$\eta(s) = \alpha_1 + \omega_1(s) + e_1(s)$$

- the marks $\mathbf{m}_1$ depend on the pattern through a joint spatially structured effect

$$\kappa(s) = \alpha_2 + \beta_1 \cdot \omega(s) + e_2(s),$$

- the marks $\mathbf{m}_2$ depend both on the spatial pattern through a joint spatial effect and on the marks $\mathbf{m}_1$

$$\nu(s) = \alpha_3 + \beta_2 \cdot \omega(s) + \beta_3 \cdot m_1(s) + e_3(s),$$

joint models

we can also fit

- **joint models** to two (or more) spatial patterns (accounting for shared environmental preferences)
- joint models of covariates AND the pattern (accounting for measurement error)
- joint models of replicated patterns
- spatio-temporal model of several species
- general measurement error models
- models for preferential sampling...

$\Rightarrow$ INLA with multiple likelihoods

reintroduction of cranes into the UK – background


BIRDGUIDES FIRST FOR BIRD NEWS

14/12/2018 Share

British crane population soars to new heights

Common Crane's remarkable British comeback continues, with a record-breaking 54 pairs counted across the country in 2018.

The total population is now believed to be in excess of 180 birds – the highest number since the species returned to Britain in 1979 after an absence of more than 400 years.

Standing at a height of 120 cm, Common Crane is the tallest bird found in Britain. Wild cranes were once widespread, but became extinct through hunting and the loss of their wetland habitat in the 1600s.

In 1979, a small number of wild cranes returned to East Anglia, establishing themselves in the Norfolk Broads. Thanks to conservation efforts, cranes have slowly spread to other areas of eastern England, with the species benefiting from work to improve habitat at RSPB reserves such as Lakenheath Fen and Nene Washes, as well as Natural England's Humberhead Peatlands.

- UK resident population extirpated in the 16th century
- re-established by a single breeding pair in 1979 through immigration from mainland Europe
- population boosted by further immigrants and a reintroduction project from 2010-2014
- only breed in wetlands...

modelling in fragmented habitat

- range expansion – which wetland habitats are likely to be colonised?
- looking at presence of the species in relation to environmental covariates alone could be misleading

⇝ some habitats may be outside of the population's current range

⇝ cranes won't nest where there is no wetland

the fancy spatio-temporal marked point pattern model

data:

- spatial point pattern reflects wetland locations
- marks: occupancy (nesting) over time (2010 - 2015)

$\rightsquigarrow$ point pattern used in order to take observation process into account

the fancy spatio-temporal marked point pattern model

joint model with two responses and likelihoods:

- wetlands: log Gaussian Cox process

the fancy spatio-temporal marked point pattern model

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- spatio-temporal probability of crane presence, conditional on the locations of wetlands (AR1)

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- shared Gaussian random field reflecting wetland intensity

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- spatio-temporal probability of crane presence, conditional on the locations of wetlands (AR1)
- shared Gaussian random field reflecting wetland intensity
- additional random field reflecting crane presence through space and time

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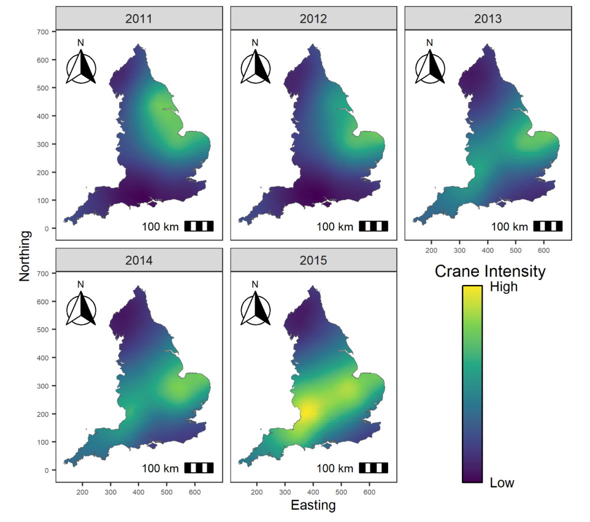
- spatio-temporal probability of crane presence, conditional on the locations of wetlands (AR1)
- shared Gaussian random field reflecting wetland intensity
- additional random field reflecting crane presence through space and time

$\rightsquigarrow$ distinguishing between unfavourable habitat and favourable habitat not within reach yet

Soriano-Redondo et al. 2019, Laxton et al. 2023

Reintroduction of cranes into the UK – some results

predictions in space and time...



suitable model...?

potential insight here:

- behaviour in near future ✓

suitable model...?

potential insight here:

- behaviour in near future ✓
- behaviour across years ✓ [not shown here]

suitable model...?

potential insight here:

- behaviour in near future ✓
- behaviour across years ✓ [not shown here]
- behaviour in more distant future...

and so...?

- current model constrains spread to southern parts of England

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- complex space-time model necessary – but very little data

$\rightsquigarrow$ joint model – wetlands and crane presence

and so...?

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- no inherent idea of spread in the models
- complex space-time model necessary – but very little data

$\rightsquigarrow$ joint model – wetlands and crane presence

$\rightsquigarrow$ point pattern reflects **observation process**

joint modelling **not** only relevant for point processes!

other examples of joint models

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there are many other scenarios where is is relevant

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- multi-species models

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- models with replicates

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joint modelling **not** only relevant for point processes!

there are many other scenarios where is is relevant

- models with different observation processes
- multi-species models
- models with replicates
- models on different (overlapping) spatial domains – with different resolutions

joint modelling: climate data – Philippines

combining data from different sources

- 57 weather stations across the > 7000 islands forming the Phillippines

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- Global Spectral Model (GSM), a numerical weather prediction model maintained by the Japan Meteorological Agency

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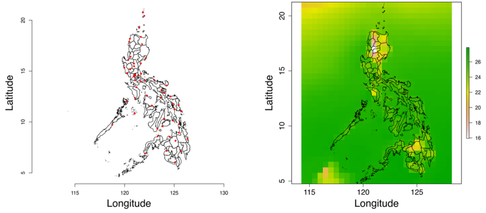

(a) Weather synoptic stations

(b) Sample GSM outcomes

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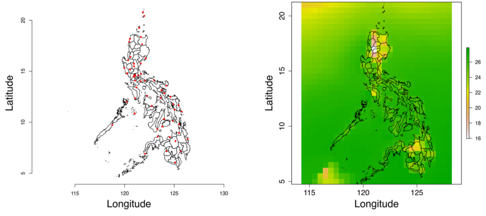
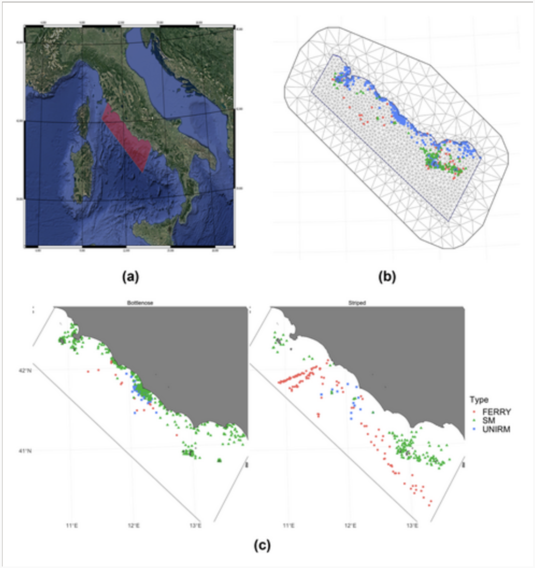


(a) Weather synoptic stations (b) Sample GSM outcomes

- weather stations: sparse but good quality
- weather model: good coverage, but not correct

⇝ model accounts for difference in data quality

joint modelling: dolphin data – citizen science and survey data



joint modelling: multi-species data – bird data

data on:

- 3 bird species: sparrow hawk, house sparrow and collard dove
- garden bird survey
- delta–gamma model – joint modelling presence and abundance
- 6 likelihoods...
- spatio-temporal model

ROYAL STATISTICAL SOCIETY
DATA | EVIDENCE | DECISIONS

Journal of the Royal Statistical Society
Applied Statistics
Series C

Appl. Statist. (2018)
67, *Part* 3, *pp.* 705–722

A spatiotemporal multispecies model of a semicontinuous response

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Ben Swallow,
University of St Andrews and University of Warwick, Coventry, UK

Janine B. Illian
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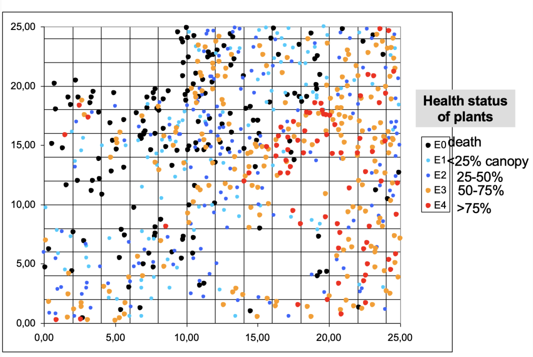
[Received January 2016. Final revision August 2017]

Summary. As accessible and potentially vulnerable species high up in the food chain, birds

joint modelling: thymus data – replicated patterns

data on:

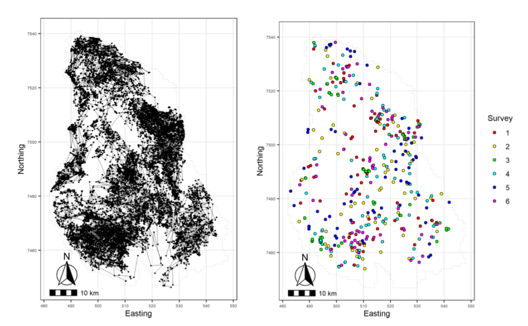
- locations of plants
- healths status of plants
- covariates: environmental pressure
- 6 replicates of the pattern



joint modelling: reindeer data – movement data and survey data

two data sources

- survey data
- GPS tracks



(a) GPS Tracks (b) Surveys

joint modelling: take-home message

- conveniently implemented in `inlabru`

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- it's fast...

joint modelling: take-home message

- conveniently implemented in `inlabru`
- it's fast...
- relevant in many contexts – not just point processes...
- a lot of scope for opportunities